

# Dendrites and conformal symmetry

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## Abstract

Progress toward characterization of structural and biophysical properties of neural dendrites together with recent findings emphasizing their role in neural computation, has propelled growing interest in refining existing theoretical models of electrical propagation in dendrites while advocating novel analytic tools. In this paper we focus on the cable equation describing electric propagation in dendrites with different geometry. When the geometry is cylindrical we show that the cable equation is invariant under the Schrödinger group and by using the dendrite parameters, a representation of the Schrödinger algebra is provided. Furthermore, when the geometry profile is parabolic we show that the cable equation is equivalent to the Schrödinger equation for the 1-dimensional free particle, which is invariant under the Schrödinger group. Moreover, we show that there is a family of dendrite geometries for which the cable equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics.

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# 1 Introduction

Dendrites are neuron branched projections first discovered by Ramon y Cajal at the end of the 19th century [1]. It is well-documented that dendrites arborize through a volume of brain tissue so as to collect information in the form of synaptic inputs. A single neuron may receive as many as 200,000 synaptic inputs through its dendrites, which account for up to 99% of a neuron’s membrane [2].

With respect to dendrite’s morphology, a trade off between the metabolic costs of dendrite elaboration and the need to cover the receptive field have been suggested as basic principles to determine their size and shape, while the connectivity between neurons has been implicated in shaping the geometry and spatial orientation of dendritic arborization [3, 4]. In this regard, novel experimental techniques enable the characterization and identification of molecules involved in different aspects of dendritic development [5].

Focusing on neural processing, theoretical studies have emphasized the role of dendrites in altering the potential range of single neuron computations while being considered as sub-units of integration with sigmoidal or Heaviside activation functions [6, 7, 8, 9, 10]. As such, these dendritic sub-units greatly expand the computational capacity of a neuron as in the case of the feature storage capacity [8], the computation of binocular disparity [11] and the computation of object-feature binding problems [12]. Also, recent experimental studies provide evidence on the role of dendritic excitability as an essential component of behaviourally relevant computations in neurons [13]. On the basis of these observations, a deeper understanding and refinement of theoretical models dealing with electrical propagation in dendrites is crucial.

The first model addressing the electrical conductivity of dendrites via the passive cable theory was formulated by Rall [14, 15]. Such a formulation remains relevant till today because passive properties of dendritic membranes provide the essential steps in the process of filtration and integration carried out by dendrites. Within this framework, a dendrite can be described as a cable with circular cross-section and diameter  $d(x)$ . Now, if  $V(x, t)$  denotes

the electrical voltage in the dendrite, the following equation is satisfied [16]

$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{1}{4r_L d(x)} \frac{\partial}{\partial x} \left( d^2(x) \frac{\partial V(x, t)}{\partial x} \right) - \frac{V(x, t)}{r_M}, \quad (1)$$

where  $c_M$  denotes the specific membrane capacitance,  $r_M$  represents the membrane resistance and  $r_L$  denotes the longitudinal resistance. In studying electric propagation in dendrites by equation (1) it is commonly assumed that their diameter is constant. However, consideration of dendrites with varying diameters is exemplified by different type of dendrite structural specializations e.g. sites of synaptic contact, also including dendrites which present varicosities as in the case of dendrites in retina amacrine cells, the cerebellar dentate nucleus and the lateral vestibular nucleus presenting as well as cortical pyramidal and olfactory bulb cells presenting bulbous enlargements on dendritic tips [2, 4].

With regard to novel mathematical techniques dealing with electric propagation in dendritic trees, the formalism of path integrals commonly derived in quantum physics has been highlighted [17]. Then, it might be possible that techniques developed in quantum physics can be employed to study electrical propagation in a dendrite. For example, quantum mathematical techniques are useful to solve differential equations. In fact, it is well known that symmetry methods in quantum physics enable defining the properties of a system without the need to solve all the equations involved in its description. In spite of this, symmetry methods have barely been applied to study differential equations that describe biological systems. It is worth mentioning that an important symmetry in physics is given by the conformal symmetry. This symmetry appears in systems as non-relativistic free particle [25, 26], atomic models [20], quark models [24], black-holes [21] and string theory [27]. Below we show that this symmetry appears in the equation (1) for different dendrite geometries.

In this paper we consider a more realistic family of dendrites that includes the types mentioned previously. Specifically, we consider dendrites with diameter

$$d(x) = d_0(1 + ax)^\nu, \quad (2)$$

where  $\nu$ ,  $d_0$ , and  $a$  are real constants. Note that different dendritic geometries can be obtained by changing the parameter  $\nu$ , for example if  $\nu = 0$ ,

the dendrite geometry is cylindrical, while if  $\nu = 1$  the dendrite geometry is conical. First, we show that when  $\nu = 0$  the cable equation (1) is invariant under the Schrödinger group, which is a conformal group, and using the dendrite parameters a representation of the Schrödinger algebra is obtained. Second, we show that when  $\nu = 2$  the cable equation is equivalent to the Schrödinger equation for the 1-dimensional free particle, which is invariant under the Schrödinger group. Third, we study the cable equation when  $\nu \neq 0$  and  $\nu \neq 2$ , in this case we show that the cable equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics. In addition, in this last case, using the dendrite parameters a representation of the conformal algebra is obtained.

This paper is organized as follows: in section 2 we provide a brief overview of the free particle and conformal quantum mechanics. In section 3, we study the particular case of cylindrical dendrites. In section 4, we address the case of dendrites with parabolic profile. In section 5, we study the relationship between dendrites and conformal quantum mechanics. In section 6, a summary is provided.

## 2 Non-relativistic conformal symmetry

In this section we provide a brief overview about the one dimensional non-relativistic free particle and the conformal quantum mechanics.

### 2.1 Free particle

The action for the 1-dimensional free non-relativistic particle is given by

$$S = \int dt \frac{m}{2} \left( \frac{dz}{dt} \right)^2, \quad (3)$$

where  $m$  is the particle mass. Note that if we take the conformal transformations

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad z' = \frac{lz + vt + c}{\gamma t + \delta}, \quad l^2 = \alpha\delta - \beta\gamma \neq 0, \quad (4)$$

where  $\alpha, \beta, \gamma, \delta, l, v, c$  are constants, then the action (3) is transformed as

$$S' = \int dt' \frac{m}{2} \left( \frac{dz'}{dt'} \right)^2 = S + \frac{m}{2} \int dt \left( \frac{d\phi(z, t)}{dt} \right), \quad (5)$$

where

$$\phi(z, t) = \frac{1}{l^2} \left( 2lvz + v^2t - \frac{\gamma(lz + vt + c)^2}{\gamma t + \delta} \right). \quad (6)$$

Then, the free non-relativistic particle dynamics is invariant under the conformal transformations (4). Such transformations (4) include temporal and spatial translations, Galileo's transformations, anisotropic scaling and the special conformal transformations.

In quantum mechanics, the wave function for the one dimensional non-relativistic free particle satisfies the Schrödinger equation

$$i\hbar \frac{\partial \Psi(z, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(z, t)}{\partial z^2}. \quad (7)$$

This equation is invariant under the conformal coordinate transformations (4), where the wave function transforms as

$$\Psi'(z', t') = \left( \sqrt{\gamma t + \delta} \right) e^{\frac{im}{2\hbar} \phi(z, t)} \Psi(z, t), \quad (8)$$

and  $\phi(z, t)$  is given by (6).

Furthermore, the following operators

$$\hat{P} = -i\hbar \frac{\partial}{\partial z}, \quad (9)$$

$$\hat{H} = \frac{\hat{P}^2}{2m}, \quad (10)$$

$$\hat{G} = t\hat{P} - mz, \quad (11)$$

$$\hat{K}_1 = t\hat{H} - \frac{1}{4}(z\hat{P} + \hat{P}z), \quad (12)$$

$$\hat{K}_2 = t^2\hat{H} - \frac{t}{2}(z\hat{P} + \hat{P}z) + \frac{m}{2}z^2. \quad (13)$$

are generators of the transformations (4). These operators satisfy the Schrödinger algebra

$$[\hat{P}, \hat{H}] = 0, \quad (14)$$

$$[\hat{P}, \hat{K}_1] = \frac{i\hbar}{2} \hat{P}, \quad (15)$$

$$[\hat{P}, \hat{K}_2] = i\hbar \hat{G}, \quad (16)$$

$$[\hat{P}, \hat{G}] = i\hbar m, \quad (17)$$

$$[\hat{H}, \hat{K}_1] = i\hbar \hat{H}, \quad (18)$$

$$[\hat{H}, \hat{G}] = i\hbar \hat{P}, \quad (19)$$

$$[\hat{H}, \hat{K}_2] = 2i\hbar \hat{K}_1, \quad (20)$$

$$[\hat{K}_1, \hat{K}_2] = i\hbar \hat{K}_2, \quad (21)$$

$$[\hat{K}_1, \hat{G}] = \frac{i\hbar}{2} \hat{G}, \quad (22)$$

$$[\hat{K}_2, \hat{G}] = 0. \quad (23)$$

It is possible to show that the operators (14)-(23) are conserved. This algebra is the so-called Schrödinger algebra. The conformal symmetry for the free Schrödinger equation was found by Niederer and Hagen in 1972 [26, 25].

In what follows we show that the conformal symmetry and the Schrödinger algebra are present in the cable equation when the dendrite geometry is cylindrical or parabolic.

## 2.2 Conformal quantum mechanics

An interesting system in quantum mechanics is given by the so-called conformal quantum mechanics [18]. This system appears in different contexts, from black-holes to atomic physics and quark models [19, 20, 21].

The Schrödinger equation for the 1-dimensional conformal quantum mechanics is

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \hat{H} \psi(x, t), \quad \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{g}{x^2}. \quad (24)$$

Here  $g$  denotes a coupling constant. When  $g \neq 0$  the Galileo's and translation symmetries are broken. Then, this system is not invariant under all conformal transformations (4). In fact, the conformal quantum mechanics is

only invariant under the coordinate transformations

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad x' = \frac{lx}{\gamma t + \delta}, \quad l^2 = \alpha\delta - \beta\gamma \neq 0. \quad (25)$$

In this case, in order to keep the wave equation invariant under the transformations (25), the wave function must be transformed as

$$\psi'(x', t') = \sqrt{\gamma t + \delta} e^{i \frac{m}{2\hbar} \Phi(x, t)} \psi(x, t) \quad (26)$$

where

$$\Phi(x, t) = -\frac{\gamma x^2}{\gamma t + \delta}. \quad (27)$$

Moreover, the generators are given by  $\hat{H}$  and

$$\hat{K}_1 = t\hat{H} - \frac{1}{2} \left( x\hat{P} - \frac{i}{2} \right), \quad (28)$$

$$\hat{K}_2 = t^2\hat{H} - t \left( x\hat{P} - \frac{i}{2} \right) + \frac{mx^2}{2} \quad (29)$$

and the algebra

$$[\hat{H}, \hat{K}_1] = i\hbar\hat{H}, \quad [\hat{H}, \hat{K}_2] = 2i\hbar\hat{K}_1, \quad [\hat{K}_1, \hat{K}_2] = i\hbar\hat{K}_2 \quad (30)$$

is satisfied. Using this algebra, it is possible to show that the operator  $\hat{H}, \hat{K}_1, \hat{K}_2$  is conserved.

Now, we show that the Schrödinger equation for the conformal quantum mechanics (24) is equivalent to the cable equation (1) for different dendrite geometries.

### 3 Cylindrical Case

If a dendrite has cylindrical geometry, the diameter is given by  $d(x) = d_0 = \text{constant}$ . In this case the cable equation (1) becomes

$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{d_0}{4r_L} \frac{\partial^2 V(x, t)}{\partial x^2} - \frac{V(x, t)}{r_M}. \quad (31)$$

The solution for this equation can be found in [16].

Notice that if we take

$$V_{cil}(x, t) = e^{-\frac{t}{c_M r_M}} \psi(x, t), \quad (32)$$

we arrive to

$$\frac{\partial \psi(x, t)}{\partial t} = \frac{d_0}{4r_L c_M} \frac{\partial^2 \psi(x, t)}{\partial x^2}. \quad (33)$$

This equation is a Schrödinger-like equation (7). Then, due that the Schrödinger equation (7) is invariant under the conformal transformations (4), the equation (33) is invariant under the same transformations. In this case the function  $\psi(x, t)$  becomes

$$\psi'(x', t') = \left( \sqrt{\gamma t + \delta} \right) e^{-\frac{c_M r_L}{d_0} \phi(x, t)} \psi(x, t), \quad (34)$$

where  $\phi(x, t)$  is given by (6). Then, the cable equation (31) is invariant under conformal transformations (4) where  $V(x, t)$  is transformed as

$$V'(x', t') = \left( \sqrt{\gamma t + \delta} \right) e^{\frac{-\gamma t^2 + (\alpha - \delta)t + \beta}{\delta t + \delta}} e^{-\frac{c_M r_L}{d_0} \phi(x, t)} V(x, t), \quad (35)$$

here  $\phi(x, t)$  is given by (6).

In addition, note that the equation (31) can be written as

$$-\frac{\partial V(x, t)}{\partial t} = \hat{H}V(x, t), \quad (36)$$

where

$$\hat{H} = \frac{d_0}{4r_L c_M} \hat{P}^2 + \frac{1}{c_M r_M}, \quad \hat{P} = -i \frac{\partial}{\partial x}. \quad (37)$$

We can see that this operator is the Hamiltonian for the non-relativistic free particle (10) with "mass"

$$m = \frac{2r_L c_M}{d_0}. \quad (38)$$



Furthermore, the following operators

$$\hat{P} = -i\frac{\partial}{\partial x}, \quad (39)$$

$$\hat{H}_0 = \frac{d_0}{4r_{LCM}}P^2, \quad (40)$$

$$\hat{G} = t\hat{P} - \frac{2r_{LCM}}{d_0}x, \quad (41)$$

$$\hat{K}_1 = t\hat{H} - \frac{1}{4}(x\hat{P} + \hat{P}x), \quad (42)$$

$$\hat{K}_2 = t^2\hat{H} - \frac{t}{2}(x\hat{P} + \hat{P}x) + \frac{r_{LCM}}{d_0}x^2. \quad (43)$$

are generators of the transformations (4). It can be shown that the operators (37) and (39)-(43) satisfy the Schrödinger algebra

$$[\hat{P}, \hat{H}_0] = 0, \quad (44)$$

$$[\hat{P}, \hat{K}_1] = \frac{i}{2}\hat{P}, \quad (45)$$

$$[\hat{P}, \hat{K}_2] = i\hat{G}, \quad (46)$$

$$[\hat{P}, \hat{G}] = i\frac{2r_{LCM}}{d_0}, \quad (47)$$

$$[\hat{H}_0, \hat{K}_1] = i\hat{H}_0, \quad (48)$$

$$[\hat{H}_0, \hat{G}] = i\hat{P}, \quad (49)$$

$$[\hat{H}_0, \hat{K}_2] = 2i\hat{K}_1, \quad (50)$$

$$[\hat{K}_1, \hat{K}_2] = i\hat{K}_2, \quad (51)$$

$$[\hat{K}_1, \hat{G}] = \frac{i}{2}\hat{G}, \quad (52)$$

$$[\hat{K}_2, \hat{G}] = 0. \quad (53)$$

Then, when the dendrite geometry is cylindrical, the cable equation (1) is equivalent to the Schrödinger equation for the non-relativistic free particle with "mass" given by (38). For this reason the conformal symmetry is present in this kind of dendrite geometry.

## 4 Parabolic case

If the dendrite geometry is parabolic, we have

$$d(x) = d_0 (1 + ax)^2, \quad (54)$$

where  $d_0$  and  $a$  are constants. In this case the cable equation (1) can be written as

$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{d_0 (1 + ax)^2}{4r_L} \frac{\partial^2 V(x, t)}{\partial x^2} + \frac{ad_0 (1 + ax)}{r_L} \frac{\partial V(x, t)}{\partial x} - \frac{V(x, t)}{r_M}. \quad (55)$$

Now, using the change of variable

$$1 + ax = e^z, \quad (56)$$

in the equation (55), we obtain

$$\frac{\partial V(z, t)}{\partial t} = \frac{a^2 d_0}{4c_M r_L} \frac{\partial^2 V(z, t)}{\partial z^2} + \frac{3a^2 d_0}{4c_M r_L} \frac{\partial V(z, t)}{\partial z} - \frac{V(z, t)}{c_M r_M}. \quad (57)$$

In addition, if we take

$$V(z, t) = e^{\left(-\frac{3z}{2} - \left(\frac{9a^2 d_0}{16r_L c_M} + \frac{1}{r_M c_M}\right)t\right)} \psi(z, t), \quad (58)$$

the equation (57) becomes

$$\frac{\partial \psi(z, t)}{\partial t} = \frac{a^2 d_0}{4c_M r_L} \frac{\partial^2 \psi(z, t)}{\partial z^2}, \quad (59)$$

which is a Schrödinger-like equation (7).

### 4.1 Symmetries

Now, due that the equation (59) is Schrödinger-like equation (7), it is invariant under the conformal transformations. In fact, using the change of variable (56), the coordinate transformations (4) can be written as

$$x' = \frac{1}{a} \left( (1 + ax)^{\left(\frac{l}{\gamma t + \delta}\right)} e^{\left(\frac{vt+c}{\gamma t + \delta}\right)} - 1 \right), \quad t' = \frac{\alpha t + \beta}{\gamma t + \delta}. \quad (60)$$

Through a long but straightforward calculation, it can be shown the cable equation (55) is invariant under the earlier transformations, where  $V(x, t)$  becomes

$$V'(x', t') = \left( \sqrt{\gamma t + \delta} \right) \left[ (1 + ax)^{-\left( \frac{3}{2} \frac{l}{\gamma t + \delta} + \frac{4c_M r_L v}{a^2 d_0 l} \right)} \right] e^{\Phi(x, t)} V(x, t), \quad (61)$$

here

$$\begin{aligned} \Phi(x, t) = & - \left[ \frac{3}{2} \frac{vt + c}{\gamma t + \delta} + \left( \frac{2c_M r_L v^2}{a^2 d_0 l^2} + \frac{9d_0 a^2}{16r_L c_M} + \frac{1}{r_M c_M} \right) t + \right. \\ & \left. - \frac{2c_M r_L \gamma}{a^2 d_0 l^2 (\gamma t + \delta)} (l \ln(1 + ax) + vt + c)^2 \right]. \end{aligned} \quad (62)$$

## 4.2 Schrödinger algebra

The cable equation (55) can be written as

$$- \frac{\partial V(x, t)}{\partial t} = \hat{H} V(x, t), \quad (63)$$

where

$$\hat{H} = \frac{d_0^2 (1 + ax)^2}{4r_L c_M} \hat{P}^2 - i \frac{a d_0 (1 + ax)}{r_L c_M} \hat{P} + \frac{1}{r_M c_M}, \quad \hat{P} = -i \frac{\partial}{\partial x}. \quad (64)$$

Now, notice that using the operator

$$\hat{\Pi} = \frac{(1 + ax)}{a} \hat{P} - i \frac{3}{2}, \quad (65)$$

the Hamiltonian (64) can be written as

$$\hat{H} = \frac{a^2 d_0}{4r_L c_M} \hat{\Pi}^2 + \frac{9}{16r_L c_M} + \frac{1}{r_M c_M}, \quad (66)$$

which is equivalent to the Hamiltonian for the non-relativistic free particle (10) with "mass"

$$\tilde{m} = \frac{2r_L c_M}{d_0 a^2}. \quad (67)$$

In addition we can construct the operators

$$\hat{H}_0 = \frac{a^2 d_0}{4r_L c_M} \hat{\Pi}^2, \quad (68)$$

$$\hat{G} = t\hat{\Pi} - \frac{2r_L c_M}{a^2} \ln(1 + ax), \quad (69)$$

$$\hat{K}_1 = t\hat{H}_0 - \frac{1}{2} \left( \ln(1 + ax) \hat{\Pi} + \hat{\Pi} \ln(1 + ax) \right), \quad (70)$$

$$\hat{K}_2 = t^2 \hat{H}_0 - \frac{t}{2} \left( \ln(1 + ax) \hat{\Pi} + \hat{\Pi} \ln(1 + ax) + \frac{r_L c_M}{a^2} [\ln(1 + ax)]^2 \right) \quad (71)$$

These operators are equivalent to the operators (39)-(43). In fact, using the relationship

$$[\ln(1 + ax), \hat{\Pi}] = i, \quad (72)$$

it is possible to show that the operators (65) and (68)-(71) satisfy the Schrödinger algebra

$$[\hat{\Pi}, \hat{H}_0] = 0, \quad (73)$$

$$[\hat{\Pi}, \hat{K}_1] = \frac{i}{2} \hat{\Pi}, \quad (74)$$

$$[\hat{\Pi}, \hat{K}_2] = i\hat{G}, \quad (75)$$

$$[\hat{\Pi}, \hat{G}] = i \frac{2c_M r_L}{a^2 d_0}, \quad (76)$$

$$[\hat{H}_0, \hat{K}_1] = i\hat{H}_0, \quad (77)$$

$$[\hat{H}_0, \hat{G}] = i\hat{\Pi}, \quad (78)$$

$$[\hat{H}_0, \hat{K}_2] = 2i\hat{K}_1, \quad (79)$$

$$[\hat{K}_1, \hat{K}_2] = i\hat{K}_2, \quad (80)$$

$$[\hat{K}_1, \hat{G}] = \frac{i}{2} \hat{G}, \quad (81)$$

$$[\hat{K}_2, \hat{G}] = 0, \quad (82)$$

Thus, when the dendrite geometry is parabolical, the cable equation (55) can be seen as a free Schrödinger equation (7) which a particular change of coordinates.

## 5 Dendrites and conformal quantum mechanics

In this section we study the cable equation when the dendrite diameter is given by (2). For this case the cable equation (1) is

$$\begin{aligned} \frac{\partial V(x, t)}{\partial t} = & \frac{d_0}{4r_L c_M} (1 + ax)^\nu \frac{\partial^2 V(x, t)}{\partial x^2} + \frac{d_0 \nu a}{2r_L c_M} (1 + ax)^{\nu-1} \frac{\partial V(x, t)}{\partial x} \\ & - \frac{V(x, t)}{r_M c_M}. \end{aligned} \quad (83)$$

Now, using the change of variable

$$z = \frac{(1 + ax)^{1 - \frac{\nu}{2}}}{a \left(1 - \frac{\nu}{2}\right)}, \quad (84)$$

we arrive to

$$\frac{\partial V(z, t)}{\partial t} = \frac{d_0}{4r_L c_M} \frac{\partial^2 V(z, t)}{\partial z^2} + \frac{3d_0 \nu}{8r_L c_M \left(1 - \frac{\nu}{2}\right)} \frac{1}{z} \frac{\partial V(z, t)}{\partial z} - \frac{V(z, t)}{r_M c_M}. \quad (85)$$

Notice that when  $\nu = 2$  or  $a = 0$  the change of variable (84) is singular, these cases were previously studied.

Now, if we take

$$V(z, t) = (z)^{-\frac{3\nu}{4(1 - \frac{\nu}{2})}} e^{-\frac{t}{r_M c_M}} \psi(z, t), \quad (86)$$

the following equation

$$\frac{\partial \psi(z, t)}{\partial t} = \frac{d_0}{4r_L c_M} \frac{\partial^2 \psi(z, t)}{\partial z^2} + \frac{3d_0 \nu (4 - 5\nu)}{16r_L c_M (2 - \nu)^2} \frac{1}{z^2} \psi(z, t) \quad (87)$$

is obtained. In addition, if we take

$$\hat{H} = -\frac{1}{2m_0} \frac{\partial^2}{\partial z^2} + \frac{g}{z^2}, \quad m_0 = \frac{2r_L c_M}{d_0}, \quad g = \frac{3d_0 \nu (5\nu - 4)}{16r_L c_M (2 - \nu)^2}, \quad (88)$$

the equation (87) can be written as

$$-\frac{\partial \psi(z, t)}{\partial t} = \hat{H} \psi(z, t). \quad (89)$$

Notice that the operator (88) is the Hamiltonian for the conformal quantum mechanics (24).

Note that the family of dendrite diameters (2) is associated with the family of conformal Hamiltonians (40). Furthermore, observe that for each Hamiltonian (40) we have two dendrite diameters, namely each Hamiltonian is associated with two  $\nu$  values. For example,  $\nu = 0$  represents a cylindrical dendrite while  $\nu = \frac{4}{5}$  represents a particular conical dendrite, but both give rise to the equation

$$\frac{\partial \psi(z, t)}{\partial t} = \frac{d_0}{4r_L c_M} \frac{\partial^2 \psi(z, t)}{\partial z^2}. \quad (90)$$

However, notice that the electrical voltage is not the same, as can be seen in the equation (86).

## 5.1 Symmetry

As the equation (87) is equivalent to the Schrödinger equation for the conformal quantum mechanics (24), then the equation (87) is invariant under the coordinate transformations (25). In this case the function  $\psi(z, t)$  transforms as

$$\psi'(z', t') = \sqrt{\gamma t + \delta} e^{-\frac{d_0}{r_L c_M} \Phi(z, t)} \psi(z, t). \quad (91)$$

Hence, the cable equation (83) is invariant under the coordinate transformations (25) where the voltage potential transforms as

$$V'(x', t') = \frac{(\gamma t + \delta)^{\frac{1+\nu}{2-\nu}}}{a^{\frac{3\nu}{2(2-\nu)}}} e^{-\frac{1}{r_M c_M} \left( \frac{-\gamma t^2 + (\alpha - \delta)t + \beta}{\gamma t + \delta} \right)} e^{\frac{4\gamma d_0}{a^2 r_M c_M} \frac{(1+ax)^{2-\nu}}{(\gamma t + \delta)(2-\nu)^2}} V(x, t). \quad (92)$$

## 5.2 Conformal algebra

The equation (83) can be written as

$$-\frac{\partial V(x, t)}{\partial t} = \hat{\mathbf{H}} V(x, t), \quad (93)$$

where

$$\hat{\mathbf{H}} = -\frac{d_0}{4r_L c_M} (1 + ax)^\nu \frac{\partial^2}{\partial x^2} - \frac{d_0 \nu a}{2r_L c_M} (1 + ax)^{\nu-1} \frac{\partial}{\partial x} + \frac{1}{r_M c_M}. \quad (94)$$

Now, using the operator

$$\hat{\Pi} = -i(1+ax)^{\frac{\nu}{2}} \frac{\partial}{\partial x} - i \frac{3\nu a}{4} (1+ax)^{\frac{\nu}{2}-1}, \quad (95)$$

the Hamiltonian (94) can be written as

$$\hat{\mathbf{H}} = \frac{d_0}{4r_L c_M} \hat{\Pi}^2 + \frac{3a^2 d_0}{64r_L c_M} (5\nu - 4) (1+ax)^{\nu-2} + \frac{1}{r_M c_M}. \quad (96)$$

In addition the following relation

$$\left[ \frac{(1+ax)^{1-\frac{\nu}{2}}}{a(1-\frac{\nu}{2})}, \hat{\Pi} \right] = i \quad (97)$$

is satisfied.

Then, the following operators

$$\mathbf{H}_0 = \frac{d_0}{4r_L c_M} \Pi^2 + \frac{3a^2 d_0}{64r_L c_M} (5\nu - 4) (1+ax)^{\nu-2}, \quad (98)$$

$$\hat{\mathbf{K}}_1 = t\hat{H}_0 - \frac{1}{2} \left( \frac{(1+ax)^{1-\frac{\nu}{2}}}{a(1-\frac{\nu}{2})} \hat{\Pi} - \frac{i}{2} \right), \quad (99)$$

$$\hat{\mathbf{K}}_2 = t^2 \hat{H}_0 - t \left( \frac{(1+ax)^{1-\frac{\nu}{2}}}{a(1-\frac{\nu}{2})} \hat{\Pi} - \frac{i}{2} \right) + \frac{r_L c_M}{d_0} \frac{(1+ax)^{2-\nu}}{a^2 (1-\frac{\nu}{2})^2} \quad (100)$$

satisfy the conformal algebra

$$[\hat{\mathbf{H}}_0, \hat{\mathbf{K}}_1] = i\hat{\mathbf{H}}_0, \quad [\hat{\mathbf{H}}_0, \hat{\mathbf{K}}_2] = 2i\hat{\mathbf{K}}_1, \quad [\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2] = i\hat{\mathbf{K}}_2. \quad (101)$$

### 5.3 Solution

In addition, if we take

$$\psi(z, t) = e^{-Et} \phi(z) \quad (102)$$

the equation (87) becomes

$$E\phi(z) = \hat{H}\phi(z), \quad (103)$$

The solution for the equation (103) is given by

$$\phi(z) = z^{\frac{1}{2}} \left[ AJ \sqrt{\frac{1}{4} + \frac{3\nu(5\nu-4)}{4(2-\nu)^2}} \left( \sqrt{\frac{4c_M r_L E}{d_0}} z \right) + BN \sqrt{\frac{1}{4} + \frac{3\nu(5\nu-4)}{4(2-\nu)^2}} \left( \sqrt{\frac{4c_M r_L E}{d_0}} z \right) \right],$$

where  $J_\alpha(z)$  is a Bessel function,  $N_\alpha(z)$  is a Neumann function and  $A, B$  are constants. Then, the solution for the equation (83) is given by

$$\begin{aligned} V(x, t) = & (1 + ax)^{\frac{(1+\nu)}{2}} e^{-\left(E + \frac{1}{c_M r_L}\right)t} \left[ AJ \sqrt{\frac{1}{4} + \frac{3\nu(5\nu-4)}{4(2-\nu)^2}} \left( \sqrt{\frac{4c_M r_L E}{d_0}} \frac{(1 + ax)^{\nu - \frac{1}{2}}}{a \left(\frac{\nu}{2} - 1\right)} \right) \right. \\ & \left. + BN \sqrt{\frac{1}{4} + \frac{3\nu(5\nu-4)}{4(2-\nu)^2}} \left( \sqrt{\frac{4c_M r_L E}{d_0}} \frac{(1 + ax)^{\nu - \frac{1}{2}}}{a \left(\frac{\nu}{2} - 1\right)} \right) \right]. \end{aligned} \quad (104)$$

The conformal quantum mechanics appears in different contexts, from black-holes to atomic physics and quark models [19, 20, 21]. Notably, this system describes a electrical voltage in different dendrite geometries.

## 6 Summary

In this paper we studied the cable equation and its symmetries for different dendritic geometries. When the geometry is cylindrical we showed that the cable equation is invariant under the Schrödinger group, and using the dendrite parameters a representation of the Schrödinger algebra was obtained. Furthermore, we showed that when the geometry is parabolical the cable equation is equivalent to the Schrödinger equation for the 1-dimensional free particle, which is invariant under the Schrödinger group. In addition, it was shown that the cable equation is equivalent to the Schrödinger equation for the 1-dimensional conformal quantum mechanics for a family of dendritic geometries. A generalized solution for the voltage of the considered family of dendrites is provided. From the neuroscience perspective, the considered family of dendrites include geometries that are more realistic when considering electric propagation in dendritic synapses as well as bulbous enlargements and varicosities that are present in various types of dendrites in the human brain.



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